Calculus for AP Physics-C

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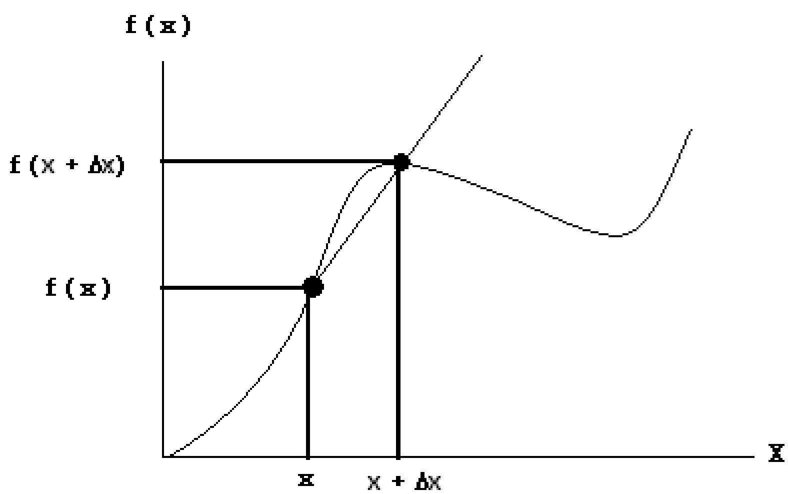
As you will quickly see, Calculus is not really a subject of AP Physics, it is a mathematical tool that we use to solve problems. However, many of you are now just starting Calculus. For that reason, this is a high speed mini-course to help you learn how to use the Calculus tools. While this material does involve some proofs (or partial proofs) most of the gruesome details are left to your math teacher.

You may not realize, but we have already done Calculus in first year Physics. Every time you found the slope of a line you were doing Differential Calculus. Every time you found the area under a curve you were doing Integral Calculus. The only differences were that we usually used real data points whereas in this class (and Calculus) we will use functions fitted to real data to find the slopes (Differential Calculus) or the areas (Integral Calculus).

Differential Calculus

Since we will be finding tangent line slopes without using real data points, we need to find a way to "sneak up" on the slopes. Tangent lines to a curve touch the curve "locally" at only one point (the point of tangency). Since we need two points to compute a slope we will use the Secant Line slope method of finding approximate slopes and then drive the secant line to a tangent line by moving one end of the secant line toward the other end.

For example, suppose we have the following graph of a function f(x):



Using basic algebra, the slope of the line through the two points $(x, f(x))$ and $(x + \Delta x, f(x + \Delta x))$ is the slope of the secant line which is (recalling that slope is rise over run)

$$\text{Slope } = \frac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x} = \frac{f(x+\Delta x)-f(x)}{\Delta x} \quad (1)$$

Now what we really want is the slope as Δx approaches zero (moving the right hand point closer and closer to the left hand point) and the secant line starts to look like the tangent line touching the function $f(x)$ at only one point "near" x. (See http://www.math.umn.edu/~garrett/qy/Secant.html for an animation.)

Suppose $f(x) = kx^2$, what is the slope of the curve at any point $(x, f(x))$? Well, let's use equation (1) to find the slope of a particular tangent line.

$$\text{Slope of } [f(x) = kx^2] = \frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{k(x+\Delta x)^2 - kx^2}{\Delta x} \quad (2)$$

Those of you who have taken Calculus know that a large part of Calculus involves "sneaky" algebraic manipulations, which often start out making things messy, and then proceed to reduce to something easy (relying on the old adage, "sometimes things have to get worse before they can get better").

There are THREE MAJOR TRICKS IN MATHEMATICS that one can do to an expression that DOES NOT change the numeric value of the expression. We will use each of these in the following derivations:

1. Expand any expression that can be expanded.
2. Add ZERO to the expression
3. Multiply the entire expression by ONE.

We will begin with method 1.

$$\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{k(x+\Delta x)^2 - kx^2}{\Delta x} = \frac{k(x^2+2x\Delta x+\Delta x^2)-kx^2}{\Delta x} = \frac{kx^2+2kx\Delta x+k\Delta x^2-kx^2}{\Delta x}$$

$$= \frac{2kx\Delta x + k\Delta x^2}{\Delta x} = 2kx + k\Delta x$$

Therefore, the slope of the secant line is $2kx + k\Delta x$. Now as Δx approaches zero the slope becomes $2kx$.

Suppose we have $f(x) = kx^3$, what is the slope of this function? Following the same procedure as before yields:

$$\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{k(x+\Delta x)^3 - kx^3}{\Delta x} = \frac{k(x^3+3x^2\Delta x+3x\Delta x^2+\Delta x^3)-kx^3}{\Delta x} = \frac{kx^3+3kx^2\Delta x+3kx\Delta x^2+\Delta x^3-kx^3}{\Delta x}$$

$$= \frac{3kx^2\Delta x+3kx\Delta x^2+\Delta x^3}{\Delta x} = 3kx^2+3kx\Delta x+\Delta x^2$$

Thus the slope of the secant line is $3kx^2+3kx\Delta x+\Delta x^2$. Now as Δx approaches zero both the second and third terms get closer and closer to zero and the slope of the tangent line becomes $3kx^2$.

Let's generalize this process and find the slope of $f(x) = kx^n$?

Well, first we have to remember how to expand an item like $(a+b)^n$ (remember the fibonacci triangle?)

$(a+b)^0 = 1$

$(a+b)^1 = 1a + 1b$

$(a+b)^2 = 1a^2 + 2ab + 1b^2 = 1a^2 + 2ab + (\text{terms with } b^2 \text{ or larger terms})$

$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3 = 1a^3 + 3a^2b + (\text{terms with } b^2 \text{ or larger terms})$

$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4 = 1a^4 + 4a^3b + (\text{terms with } b^2 \text{ or larger terms})$

$(a+b)^5 = 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5 = 1a^5 + 5a^4b + (\text{terms with } b^2 \text{ or larger terms})$

Notice that the first term is always $1a^{\text{original power}}$

and the second term is always $(\text{original power})a^{\text{original power}-1}b$

and the rest of the terms have $b^{\text{power}>1}$

Now, what is $(x+\Delta x)^n$?
$(x+\Delta x)^n = x^n + nx^{n-1}\Delta x + (\text{bunch of terms})\Delta x^{\text{power}}$ (where power ≥ 2)

Now, lets use this idea to find the slope of *f(x) = kx^n*

$$\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{k(x+\Delta x)^n - kx^n}{\Delta x} = \frac{k(x^n + nx^{n-1}\Delta x + (\text{bunch of terms})\Delta x^{\text{power}\geq 2}) - kx^n}{\Delta x}$$

$$= \frac{kx^n + knx^{n-1}\Delta x + k(\text{bunch of terms})\Delta x^{\text{power}\geq 2} - kx^n}{\Delta x}$$

$$= \frac{knx^{n-1}\Delta x + k(\text{bunch of terms})\Delta x^{\text{power}\geq 2}}{\Delta x} = nkx^{n-1} + k(\text{bunch of terms})\Delta x^{\text{power}-1}$$

But in the last power, the phrase "$\text{power} - 1$" is still ONE or bigger since the original power ≥ 2.

The slope of $f(x) = kx^n = nkx^{n-1}$, since $k(\text{bunch of terms})\Delta x^{\text{power}-1} \to 0 \; as \; \Delta x \to 0$ (3)

What this means is that for any simple power function, the slope is simply the original function times the original power and then the power is dropped by ONE. This is a derivative. It is that easy!

Example
Suppose you are moving so that your position is given by the function $x(t) = 3t^4$. Then your velocity (which is the slope of the position curve) must be $v(t) = 4(3)t^3 = 12t^3$. Your acceleration (which is the slope of the velocity) must be $a(t) = 3(12)t^2 = 36t^2$.

If you wish, you can even find the jerk (rate of change in acceleration) as the slope of acceleration curve, which is $j(t) = 2(36)t^1 = 72t$. You can even find the snap (rate of change in jerk) as the slope of jerk curve, which is $s(t) = 1(72)t^0 = 72$, but we digress.

Notice that last step. This rule works for ANY power function with any power (positive, negative or decimal).

More Examples
$y = f(x) = 2.4x^{4.5}$
$\text{Slope} = 4.5(2.4)x^{3.5} = 10.8x^{3.5}$
$y = f(x) = -4.2x^{0.5}$
$\text{Slope} = 0.5(-4.2)x^{-0.5} = -2.1x^{-0.5}$
$y = f(x) = 11.34x^{-3.4}$
$\text{Slope} = (-3.4)(11.34)$

This means that the graphical physics we did last year can now be done very easily with a simple formula. However (there is always a catch, isn't there!?!), not all functions are of the simple form $f(x) = kx^n$. So, we need some ways to address other functions. The following describes how we will address more complex functions.

Sum Rule

Suppose we have a function that is the sum of two functions?

$y = f(x) = g(x) + h(x)$ is the standard way of saying the sum of two functions.

An example might be, $f(x) = 3x^2 + 4x^3$.

We now have to ask what $y = f(x) = g(x) + h(x)$ really means. Well, what does $y = f(x)$ mean? A true mathematician would say it is a set of numeric pairs where each unique first number has EXACTLY one second number. Or they would say that it is a way to match up to sets so that each member of the first set (domain) is connected to exactly ONE member of the second set (range). For the purposes of Physics, we restrain the sets to be the REAL NUMBERS and the connection rule is ALWAYS an Algebraic Statement (well at least in most of physics).

Thus $y = f(x)$ means put the number 'x' into the algebraic rule called 'f' and spit out its one and only value called 'y'. Above, in $f(x) = 3x^2 + 4x^3$, we compute two separate terms (independently) and then sum the results. In the case above we could have factored x^2 first to give $f(x) = x^2(3 + 4x)$ but why make it messy by turning a simple sum into a multiplication!!

When you have a sum of two unknown functions one MUST determine each separately AND THEN add.

Thus if $f(x) = g(x) + h(x)$ one must determine $g(x)$ and $h(x)$ separately and then sum their results.

The slope equation (1) becomes, Slope $= \frac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x} = \frac{[g(x+\Delta x)+h(x+\Delta x)]-[g(x)+h(x)]}{\Delta x}$

or, rearranging (invoking Rule 1 from page 2), we get $\frac{[g(x+\Delta x)+h(x+\Delta x)]-[g(x)+h(x)]}{\Delta x} = \frac{[g(x+\Delta x)-g(x)]+[h(x+\Delta x)-h(x)]}{\Delta x}$

Rearranging further, we get $\frac{[g(x+\Delta x)-g(x)]+[h(x+\Delta x)-h(x)]}{\Delta x} = \frac{[g(x+\Delta x)-g(x)]}{\Delta x} + \frac{[h(x+\Delta x)-h(x)]}{\Delta x}$

The two fractions on the right are just the slopes of function g(x) and h(x) added together.

Thus, if a function is the sum of two functions then its slope is just the sum of the two slopes.

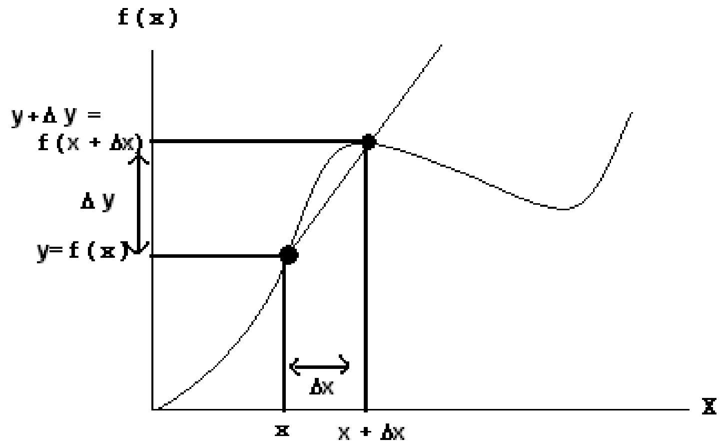
In our example, from just above

$f(x) = 3x^2 + 4x^3$.
The slope is simply $6x + 12x^2$.

[Terminology]

We are going to get tired of constantly saying "the slope of f(x) is" so let's create a shorthand way of saying the same thing.

Say we have the function $y = f(x)$.



Then at $(x + \Delta x)$, $(y + \Delta y) = f(x + \Delta x)$

Thus the slope is $\frac{(y+\Delta y)-y}{(x+\Delta x)-x} = \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$

Of course, we are asking what happens to these ratios when Δx approaches zero. The astute will notice that when Δx approaches zero, so does Δy and we end up with the famous $0/0$ conundrum.

The sneaky thing that happens is that both top and bottom go to zero together but the ratio stays the same as they approach zero. We express this "approaching zero" constant ratio as simply the slope of the tangent line. So we write the slope as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = slope \qquad (4)$$

We also denote the slope of $y = f(x)$ *as* $f'(x)$ and is spoken as "f prime of x". Thus we have for the slope of $y = f(x)$ the notation $\frac{dy}{dx} = f'(x)$

The act of finding the slope is called "differentiation" and the result of differentiation is called the "derivative".

The act of finding the slope of a line tangent to a curve is called *differentiation*. The result, which is the slope of the tangent line, is called the *derivative*.

If one wants to find a *derivative*, then one *differentiates* a function according to the prescribed rules. Repeating the examples from above using the new notation:

$y = f(x) = 2.4x^{4.5}$

$$\frac{dy}{dx} = f'(x) = 4.5(2.4)x^{3.5} = 10.8x^{3.5}$$

$y = f(x) = -4.2x^{0.5}$

$$\frac{dy}{dx} = f'(x) = (0.5)(-4.2)x^{-0.5} = -2.1x^{-0.5}$$

$y = f(x) = 11.34x^{-3.4}$

$$\frac{dy}{dx} = f'(x) = (-3.4)(11.34)x^{-4.4} = -38.556x^{-4.4}$$

What is important to realize is that all we are doing is finding the slopes of tangent lines to curves.

End of [Terminology]

Sum or Difference Rule

What happens if we have a function, which is the sum or difference of two functions?

$$y = f(x) = g(x) \pm h(x)$$

Doing the differentiation we get

$$\frac{dy}{dx} = f'(x) = \frac{f(x+\Delta x) \pm f(x)}{(x+\Delta x) - x} = \frac{[g(x+\Delta x) \pm h(x+\Delta x)] - [g(x) \pm h(x)]}{\Delta x}$$

$$= \frac{[g(x+\Delta x) - g(x)] \pm [h(x+\Delta x) - h(x)]}{\Delta x} = \frac{[g(x+\Delta x) - g(x)]}{\Delta x} \pm \frac{[h(x+\Delta x) - h(x)]}{\Delta x}$$

Note that the last two terms are simply the derivatives of $g(x)$ and $h(x)$.

Thus the derivative of the sum or difference of two functions is simply the sum or difference of the derivatives!

Therefore, we have the following:

If $\boldsymbol{y = f(x) = h(x) + g(x)}$, then $\boldsymbol{\frac{dy}{dx} = f'(x) = h'(x) + g'(x)}$ (5)

If $\boldsymbol{y = f(x) = h(x) - g(x)}$, then $\boldsymbol{\frac{dy}{dx} = f'(x) = h'(x) - g'(x)}$ (6)

Warning: the above rule does NOT work for products, quotients and compositions. We will now grind our way through these other operations.

Product Rule

Suppose we have the function

$$y = f(x) = g(x)h(x)$$

Just in case you think these functions don't exist consider a rail coal car that is having both its speed and mass change as a function of time. Momentum depends on mass and velocity, but both m and v are functions of t so the correct equation for momentum is $p = m(t)\,v(t)$, which is the product of two functions. We will see that the time derivative of momentum is the technical definition of force.

So, that being said, how do we find the slope (derivative) of a product of two functions?

$$\frac{dy}{dx} = f'(x) = \frac{g(x+\Delta x)h(x+\Delta x) - g(x)h(x)}{\Delta x}$$

Well, there it is, but what to do with it? Let's invoke RULE 2 from page 2 simply add zero to it in the following strange manner:

$$\frac{dy}{dx} = f'(x) = \frac{g(x+\Delta x)h(x+\Delta x) - g(x)h(x)}{\Delta x} = \frac{g(x+\Delta x)h(x+\Delta x) - g(x)h(x) + g(x+\Delta x)h(x) - g(x+\Delta x)h(x)}{\Delta x}$$

As strange as that seems, it is logically legal since adding zero does not change the expression on the right side of the equality thus the equality is still true. The last two terms are the same number except one is positive and the other is negative. Now, rearranging the mess above we get

$$\frac{dy}{dx} = f'(x) = \frac{g(x+\Delta x)h(x+\Delta x) - g(x)h(x) + g(x+\Delta x)h(x) - g(x+\Delta x)h(x)}{\Delta x}$$

$$= \frac{g(x+\Delta x)h(x+\Delta x) - g(x+\Delta x)h(x) + g(x+\Delta x)h(x) - g(x)h(x)}{\Delta x}$$

$$= \frac{g(x+\Delta x)[h(x+\Delta x) - h(x)] + [g(x+\Delta x) - g(x)]h(x)}{\Delta x}$$

$$= \frac{g(x+\Delta x)[h(x+\Delta x) - h(x)]}{\Delta x} + \frac{[g(x+\Delta x) - g(x)]h(x)}{\Delta x}$$

$$= g(x+\Delta x)\frac{[h(x+\Delta x) - h(x)]}{\Delta x} + \frac{[g(x+\Delta x) - g(x)]}{\Delta x}h(x)$$

$= g(x)h'(x) + g'(x)h(x)$ as $\Delta x \to 0$

Thus, the derivative of $y = f(x) = g(x)h(x)$ is

$$\boldsymbol{\frac{dy}{dx} = f'(x) = g(x)h'(x) + g'(x)h(x)} \tag{7}$$

As an example, consider the function with $g(x) = 3x^2$ and $h(x) = 4x^3$

$y = g(x)h(x) = (3x^2)(4x^3) = 12x^5$

This is a silly example since one can just multiply the two functions and then do the derivative.

If $y = 12x^5$ then $\frac{dy}{dx} = 60x^4$ from what we have already learned.

However, using the new rule, we get

$\frac{dy}{dx} = g(x)h'(x) + g'(x)h(x) = (3x^2)(12x^2) + (6x)(4x^3) = (36x^4) + (24x^4) = 60x^4$, which is the same.

Warning: DO NOT MULTIPLY THE DERIVATIVES.

If we multiply the derivatives in the previous example, then we would erroneously get
$\frac{dy}{dx} = (6x^1)(12x^2) = 72x^3$ **THIS IS WRONG – SO DON'T DO IT.**

The Rule for differentiating functions multiplied together is:
The FIRST times the derivative of the SECOND plus the derivative of the FIRST times the SECOND.

Quotient Rule

What happens when you have one function divided by another?

$$y = f(x) = \frac{g(x)}{h(x)}$$

$$\frac{dy}{dx} = f'(x) = \frac{\frac{g(x+\Delta x)}{h(x+\Delta x)} - \frac{g(x)}{h(x)}}{\Delta x}$$

Now, to add (or subtract) fractions, one needs a common denominator. But, this common denominator is in the numerator of the larger fraction. Thus I will call this number the "common denumerator" – to coin a phrase.

$$\frac{dy}{dx} = f'(x) = \frac{\frac{g(x+\Delta x)}{h(x+\Delta x)} - \frac{g(x)}{h(x)}}{\Delta x} = \frac{\frac{g(x+\Delta x)h(x) - g(x)h(x+\Delta x)}{h(x+\Delta x)h(x)}}{\Delta x} = \frac{g(x+\Delta x)h(x) - g(x)h(x+\Delta x)}{\Delta x\, h(x+\Delta x)h(x)}$$

$$= \frac{1}{h(x+\Delta x)h(x)} * \frac{g(x+\Delta x)h(x) - g(x)h(x+\Delta x)}{\Delta x}$$

Now, I will do the magic and add ZERO again. If you have to ask why I chose this particular ZERO then you have the makings of a mathematician. The answer is: After months (years) of trial and error Newton (Leibnitz?) found what works!!

$$\frac{dy}{dx} = f'(x) = \frac{1}{h(x+\Delta x)h(x)} * \frac{g(x+\Delta x)h(x) - g(x)h(x+\Delta x)}{\Delta x}$$

$$= \frac{1}{h(x+\Delta x)h(x)} * \frac{g(x+\Delta x)h(x) - g(x)h(x+\Delta x) + g(x)h(x) - g(x)h(x)}{\Delta x}$$

$$= \frac{1}{h(x+\Delta x)h(x)} * \frac{g(x+\Delta x)h(x) - g(x)h(x) - g(x)h(x+\Delta x) + g(x)h(x)}{\Delta x}$$

$$= \frac{1}{h(x+\Delta x)h(x)} * \frac{h(x)\left[g(x+\Delta x) - g(x)\right] - g(x)\left[h(x+\Delta x) - h(x)\right]}{\Delta x}$$

Please note the factorization in the last line. Read it carefully.

$$\frac{dy}{dx} = f'(x) = \frac{1}{h(x+\Delta x)h(x)} * \frac{h(x)\left[g(x+\Delta x) - g(x)\right] - g(x)\left[h(x+\Delta x) - h(x)\right]}{\Delta x}$$

$$= \frac{1}{h(x+\Delta x)h(x)} * \left[\frac{h(x)\left[g(x+\Delta x) - g(x)\right]}{\Delta x} - \frac{g(x)\left[h(x+\Delta x) - h(x)\right]}{\Delta x}\right]$$

$$= \frac{1}{h(x)h(x)} * \left[h(x)g'(x) - h'(x)g(x)\right] = \frac{h(x)g'(x) - h'(x)g(x)}{h(x)h(x)}$$

What this says is that the derivative of a ratio of two functions is:

The BOTTOM times the derivative of the TOP minus the derivative of the BOTTOM times the TOP, ALL divided by the BOTTOM squared.

Here is a way of memorizing this technique:

Suppose $f(x) = h_i/h_o$ (High over Hoe)

Then the derivative is Ho deeHi – Hi deeHo Over HoHo

The Chain Rule

Suppose you have a function of a function. In other words, compute a function then use the result of that computation to compute a second function. It sounds uncommon but it is very common. Here are some examples:

$y = sin(2x)$ (The 2x must be computed first then the sin computed)

$y = (x^2 + 1)^3$ (Normally, one would compute the $x^2 + 1$ first then cube the result though it could be expanded)

Now, suppose we have the following:

$y = f(x) = g(h(x))$ which simply means compute $h(x)$ then use the result to compute the 'g' function.

What is the derivative of $f(x)$?

$$\frac{dy}{dx} = f'(x) = \frac{g(h(x+\Delta x)) - g(h(x))}{\Delta x}$$

Now, what does one do with this? Well the trick is to use RULE 3 from page 2 and multiply by ONE.

$$\begin{aligned}\frac{dy}{dx} = f'(x) &= \frac{g(h(x+\Delta x)) - g(h(x))}{\Delta x}\,\frac{h(x+\Delta x) - h(x)}{h(x+\Delta x) - h(x)} \\ &= \frac{g(h(x+\Delta x)) - g(h(x))}{h(x+\Delta x) - h(x)} * \frac{h(x+\Delta x) - h(x)}{\Delta x} \\ &= \frac{g(h(x+\Delta x)) - g(h(x))}{h(x+\Delta x) - h(x)} * h'(x)\end{aligned}$$

Now we have to get sneaky and name

$z = h(x)$ which also implies that $z + \Delta z = h(x + \Delta x)$ giving us

$$\begin{aligned}\frac{dy}{dx} = f'(x) &= \frac{g(h(x+\Delta x)) - g(h(x))}{h(x+\Delta x) - h(x)} * h'(x) \\ &= \frac{g(z+\Delta z) - g(z)}{(z+\Delta z) - z} * h'(x) \\ &= \frac{g(z+\Delta z) - g(z)}{\Delta z} * h'(x) \\ &= g'(z) * h'(x) \\ &= g'(h(x)) * h'(x)\end{aligned}$$

What the alphabet soup above means is that one takes the derivative of the 'outer' function and just substitutes in the 'inner' function, then multiplies the whole thing by the derivative of the inner function.

Let's look at some examples:

First let's use the second example from the start of the Chain Rule which was

$y \ = \ f(x) \ = \ g(h(x)) \ = \ (x^2+1)^3$

This is a combination of

$h(x) = x^2 + 1$ and $g(x) = x^3$

$g'(x) = 3x^2$ or $g'(h(x)) = 3(h(x))^2 = 3(x^2+1)^2$

and $h'(x) = 2x + 0 = 2x$

Therefore,

$$\frac{dy}{dx} = f'(x) = g'(h(x))h'(x) = 3(x^2+1)^2(2x) = 6x(x^2+1)^2 = 6x(x^4+2x^2+1) = 6x^5+12x^3+6x$$

Now let's show that this is right by doing the derivative another way. The original function was $y \ = f(x) = (x^2+1)^3 = x^6 + 3x^4 \ + 3x^2 + 1$ when expanded out.

$$\frac{dy}{dx} = f'(x) = 6x^5 + 12x^3 + 6x$$

Both derivatives are the same.

Multiplying a function by a constant:
I have left this item to last simply because I forgot to do it earlier!

Suppose we have the function $y = k \cdot f(x) = kf(x)$ where $k = \text{constant}$. What is its derivative?

$$y = k\ f(x), \quad therfore\ y' = \frac{dy}{dx} = \frac{k(f(x+\Delta x) - kf(x)}{\Delta x} = k\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right] = kf'(x)$$

Well, that was easy. The derivative of a constant times a function is the constant times the derivative of the function.

Before you can use Calculus you will need to do LOTS of practice taking derivatives of many functions before you are ready to use Calculus as a tool in Physics and other areas.

Review of what we have learned so far (yes there is more to learn).

If $y = kx^n$ then $\frac{dy}{dx} = nkx^{n-1}$. This is the ONLY function specific rule – so far.

If $y = f(x) + g(x)$ then $\frac{dy}{dx} = f'(x) + g'(x)$

If $y = f(x) - g(x)$ then $\frac{dy}{dx} = f'(x) - g'(x)$

If $y = f(x)g(x)$ then $\frac{dy}{dx} = f(x)g'(x) + f'(x)g(x)$

If $y = f(x)/g(x)$ then $\frac{dy}{dx} = (g(x)f'(x) - f(x)g'(x))/(g(x))^2$

If $y = f(g(x))$then $\frac{dy}{dx} = f'(g(x))g'(x)$

If $y = kf(x)$ ($k =$ constant) then $\frac{dy}{dx} = kf'(x)$

Practice will make you perfect – EVENTUALLY!

Outside of polynomials, there are as many specialty (non-polynomial) functions as there are grains of sand in the universe. We will concern ourselves with a very useful (and small) subset of these functions. They are:

$y = sin(x)$

$y = cos(x)$

$y = ax$

$y = e^x$

$y = ln(x)$

$y = sin(x)$

Let's find the derivative of $y = sin(x)$.

$$\frac{dy}{dx} = \frac{\sin(x + \Delta x) - \sin(x)}{\Delta x}$$

Now what to do? The trick is to convert the left term in the numerator using a trig identity that we all know and remember with fondness. If you are not in that group that remembers such stuff, then get a math book out and look up trig identities.

I will use the trig. identity: $sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$.

$$\frac{dy}{dx} = \frac{\sin(x + \Delta x) - \sin(x)}{\Delta x} = \frac{\sin(x)\cos(\Delta x) + \sin(\Delta x)\cos(x) - \sin(x)}{\Delta x}$$

Now, slightly rearrange and then do a bit of factoring.

$$\frac{dy}{dx} = \frac{\sin(x)\cos(\Delta x)+\sin(\Delta x)\cos(x) - \sin(x)}{\Delta x} = \frac{\sin(x)\left[\cos(\Delta x) - 1\right]+\sin(\Delta x)\cos(x)}{\Delta x}$$

$$= \frac{\sin(x)\left[\cos(\Delta x) - 1\right]}{\Delta x} + \frac{\sin(\Delta x)\cos(x)}{\Delta x} = \sin(x)*\frac{\left[\cos(\Delta x) - 1\right]}{\Delta x} + \cos(x)*\frac{\sin(\Delta x)}{\Delta x}$$

Now it is time to bypass the Limit method you will learn (or have already seen) in Calculus and just do some arithmetic. (Note to student: this is a live Excel Chart – just double click on it if you are reading this file in MS Word©.)

Δx	Δx	cos(Δx)	cos(Δx)-1	(cos(Δx)-1)/Δx	sin(Δx)	sin(Δx)/Δx
Degrees	Radian					
45	0.7853982	0.70710678119	-0.29289321881	-0.3729232	0.7071068	0.900316316157
30	0.5235988	0.86602540378	-0.13397459622	-0.2558726	0.5000000	0.954929658551
15	0.2617994	0.96592582629	-0.03407417371	-0.1301538	0.2588190	0.988615929465
10	0.1745329	0.98480775301	-0.01519224699	-0.0870452	0.1736482	0.994930770045
5	0.0872665	0.99619469809	-0.00380530191	-0.0436055	0.0871557	0.998731243954
2	0.0349066	0.99939082702	-0.00060917298	-0.0174515	0.0348995	0.999796934092
1	0.0174533	0.99984769516	-0.00015230484	-0.0087264	0.0174524	0.999949231203
0.5	0.0087266	0.99996192306	-0.00003807694	-0.0043633	0.0087265	0.999987307656
0.25	0.0043633	0.99999048072	-0.00000951928	-0.0021817	0.0043633	0.999996826905
0.125	0.0021817	0.99999762018	-0.00000237982	-0.0010908	0.0021817	0.999999206726
0.0625	0.0010908	0.99999940504	-0.00000059496	-0.0005454	0.0010908	0.999999801681
0.03125	0.0005454	0.99999985126	-0.00000014874	-0.0002727	0.0005454	0.999999950420
0.015625	0.0002727	0.99999996282	-0.00000003718	-0.0001364	0.0002727	0.999999987605
0.007813	0.0001364	0.99999999070	-0.00000000930	-0.0000682	0.0001364	0.999999996901
0.003906	0.0000682	0.99999999768	-0.00000000232	-0.0000341	0.0000682	0.999999999225
0.001953	0.0000341	0.99999999942	-0.00000000058	-0.0000170	0.0000341	0.999999999806
0.000977	0.0000170	0.99999999985	-0.00000000015	-0.0000085	0.0000170	0.999999999952
0.000488	0.0000085	0.99999999996	-0.00000000004	-0.0000043	0.0000085	0.999999999988

The first column starts out with common angles in degrees until 1^o and then just starts halving the previous angle. The second column is the degree converted into radians. It probably is worth noting (maybe even making a big deal) that a radian is NOT a unit. It is simply a fancy way of saying ONE – the ratio of the arc length to the radius, thus its construction units cancel. In fact, we will see examples later (particularly in Simple Harmonic Motion and Rotation) where the word radian is purposely ignored and in fact deleted. By the way, 1 radian = $180^o/\pi$ = about 57.296^o.

All the other columns do Trig. with the radian value since Excel does Trig in radians.

The fifth column is the ratio $\frac{\cos(\Delta x)-1}{\Delta x}$ which approaches ZERO as $\Delta x \to 0$.

The seventh column is the ratio $\frac{\sin(\Delta x)}{\Delta x}$ which approaches ONE as $\Delta \mathrm{x} \to 0$.

Small digression before returning to Calculus:

This last fact above is very useful when we need to simplify a hard to compute problem. Having a ratio approach ONE simply implies that the numerator and denominator become the SAME number as the denominator approaches zero. Thus if one has a process that uses the sine of small angles, one can often just use the angle itself instead of the sine of the angle. Look back at the table. At 30^o, the $\sin(30^0)$ and 30^0 in radians are within 5% of each other. At 15^o the two numbers are within 1.1%. At 5^o, the numbers are within 0.1%. This means that if one is *measuring to within 5%, the angle (in radians) can be used instead of the sine(angle) up to 30^o. Errors less than 1% allow the substitution below 5^o.*

We will use this "Small Angle Conversion" when we study Simple Harmonic Motion and when we look at Interference (assuming we have the time).
End small digression. On with Calculus.

The ratios above imply the following:

$$\frac{dy}{dx} = \frac{\sin(x)\cos(\Delta x)+\sin(\Delta x)\cos(x)-\sin(x)}{\Delta x} = \frac{\sin(x)\left[\cos(\Delta x)-1\right]+\sin(\Delta x)\cos(x)}{\Delta x}$$
$$= \frac{\sin(x)\left[\cos(\Delta x)-1\right]}{\Delta x}+\frac{\sin(\Delta x)\cos(x)}{\Delta x} = \sin(x)*\frac{\left[\cos(\Delta x)-1\right]}{\Delta x}+\cos(x)*\frac{\sin(\Delta x)}{\Delta x}$$
$$= \sin(x)*ZERO+\cos(x)*ONE = \cos(x)$$

Thus the derivative of $\boldsymbol{sin(x)}$ is $\boldsymbol{cos(x)}$. Now that is a nice and easy fact!

$y = cos(x)$

Now let's find the derivative of $y = cos(x)$.

$$\frac{dy}{dx} = \frac{\cos(x+\Delta x)-\cos(x)}{\Delta x} = \frac{\cos(x)\cos(\Delta x)-\sin(x)\sin(\Delta x)-\cos(x)}{\Delta x} = \frac{\cos(x)\left[\cos(\Delta x)-1\right]-\sin(x)\sin(\Delta x))}{\Delta x}$$
$$= \cos(x)*\frac{\left[\cos(\Delta x)-1\right]}{\Delta x}-\sin(x)*\frac{\sin(\Delta x)}{\Delta x}$$

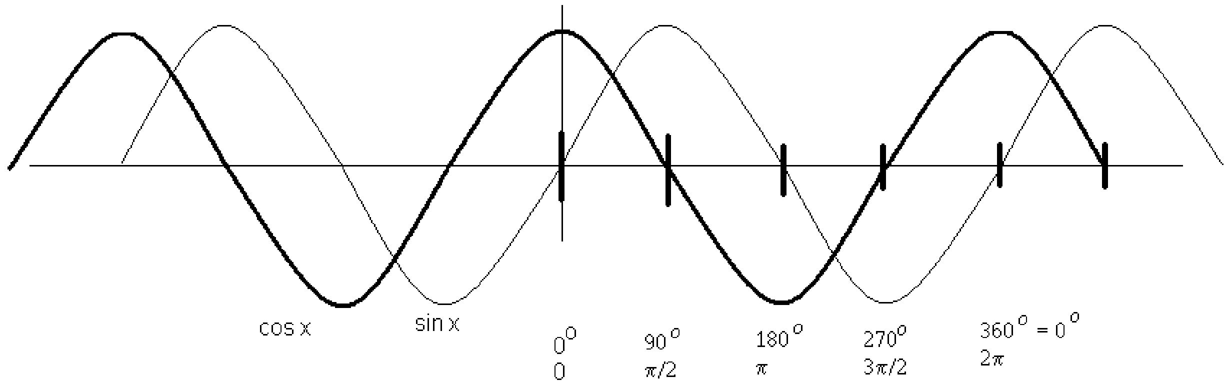
We used the Trig Identity $cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$

Those two ratios should look very familiar from above so let's just use them.

$$\frac{dy}{dx} = \cos(x)*\frac{\left[\cos(\Delta x)-1\right]}{\Delta x}-\sin(x)*\frac{\sin(\Delta x)}{\Delta x} = \cos(x)*ZERO-\sin(x)*ONE = -\sin(x)$$

Thus, the derivative of $\boldsymbol{cos(x)}$ is – $\boldsymbol{sin(x)}$.

Note that the slope numbers of the sine curve are the same numbers as the cosine curve which implies that



	0^o (0)	90^o ($\pi/2$)	180^o (π)	270^o ($3\pi/2$)	$360^o = 0^o$ (2π)
sin x =	0	1	0	-1	0
cos(x)=	1	0	-1	0	1
Graph Slope of Sin =	1	0	-1	0	1
Graph Slope of Cos=	0	-1	0	1	0

If $y = sin(x)$ then $\frac{dy}{dx} = y' = cos(x)$.

Note also that the slope numbers of cosine are the negative of the sine numbers implying that

If $y = cos(x)$ then $\frac{dy}{dy} = y' = -sin(x)$.

Without further ado, let's find the derivative of the tangent function. We don't really need to know it for AP Physics but you now have ALL the tools you need to find it.

$$y = \tan(x) = \frac{\sin(x)}{\cos(x)}, \textit{therefore the division rule}$$

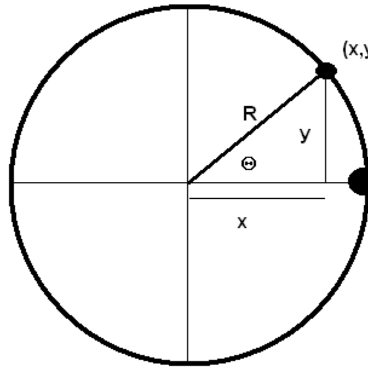
$$y' = \frac{dy}{dx} = \frac{bottom * top\ slope \ - \ top * bottom\ slope}{bottom^2} = \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{[\cos(x)]^2}$$

$$= \frac{[\cos(x)]^2 + [\sin(x)]^2}{[\cos(x)]^2} = \frac{1}{[\cos(x)]^2} = [\sec(x)]^2$$

This is just an example of how we can handle what look like hard problems by breaking them down into known chunks and applying the rules.

Now for your first major REAL PHYISCS STUFF using Calculus!

Suppose we have an object moving in a circle of radius R with a period T.



The speed of the object is simply the circumference divided by the period, thus we have $v = \frac{2\pi R}{T}$. From last year, we also know the centripetal acceleration to be

$$a_c = \frac{v^2}{R} = \frac{\left[\frac{2\pi R}{T}\right]^2}{R} = \frac{4\pi^2 R}{T^2}.$$

We did not derive this expression last year. You now know the calculus necessary to do so.

(NOTE to student: I am making the assumption that by now you have practiced doing derivatives until you are sick of them!)

If the object is moving around the circle with a velocity $2\pi R/T$ then where is it at any intermediate time 't'?

Letting $x = vt$ (Basic physics) we get $Z = \frac{2\pi R}{T}t$ where Z is length along the circumference. The central angle shown above as 'Θ' is then simply (in radians) $\Theta = \frac{Z}{R} = \frac{\frac{2\pi R}{T}t}{R} = \frac{2\pi}{T}t$.

At time t, the object has moved from $(R, 0)$ to (x, y). Now, we actually know (x, y) by doing trigonometry!

$$\frac{x}{R} = \cos(\Theta) = \cos(\frac{2\pi}{T}t)$$
$$or$$
$$x = R\cos(\frac{2\pi}{T}t)$$

Repeating for y we get

$$\frac{y}{R} = \sin(\Theta) = \sin(\frac{2\pi}{T}t)$$
$$or$$
$$y = R\sin(\frac{2\pi}{T}t)$$

Both x and y are functions of 't'. We can now actually compute how fast the object is traveling along each axis by differentiating each function (Slope of position is velocity). Note the use of the Chain Rule below!!

$$x = R\cos(\frac{2\pi}{T}t)$$

$$or$$

$$v_x = \frac{dx}{dt} = R\left[-\sin(\frac{2\pi}{T}t)\right]\frac{2\pi}{T} = -\frac{2\pi R}{T}\sin(\frac{2\pi}{T}t)$$

It may be hard to see but I have invoked THREE (3) of the differentiation rules:

1. The last rule – Constant times a function
2. The derivative of the Sine function
3. The Chain Rule

Repeating for y we get

$$y = R\sin(\frac{2\pi}{T}t)$$

$$or$$

$$v_y = \frac{dy}{dt} = R\left[\cos(\frac{2\pi}{T}t)\right]\frac{2\pi}{T} = \frac{2\pi R}{T}\cos(\frac{2\pi}{T}t)$$

Note that V_x and V_y are both components of the actual 'slant' or tangential velocity as the object moves along the circle. This means we can use the Pythagorean Theorem to calculate the actual velocity.

$$V = \sqrt{{V_x}^2 + {V_y}^2} = \sqrt{\left[-\frac{2\pi R}{T}\sin(\frac{2\pi}{T}t)\right]^2 + \left[\frac{2\pi R}{T}\cos(\frac{2\pi}{T}t)\right]^2}$$

$$= \sqrt{\left[\frac{2\pi R}{T}\right]^2\left(\sin^2(\frac{2\pi}{T}t) + \cos^2(\frac{2\pi}{T}t)\right)} = \sqrt{\left[\frac{2\pi R}{T}\right]^2(1)} = \frac{2\pi R}{T} = \frac{Circumference}{Period}$$

So, Calculus has solved a problem we already knew the answer to with far more sweat!!

But, wait, there is more. We have the x and y velocity equations. We can compute the accelerations along each axis doing exactly the same thing.

$$v_x'(t) = a_x = \frac{dv_x}{dt} = -\frac{2\pi R}{T}\cos(\frac{2\pi}{T}t)\frac{2\pi}{T} = -\frac{4\pi^2 R}{T^2}\cos(\frac{2\pi}{T}t)$$

$$and$$

$$v_y'(t) = a_y = \frac{dv_y}{dt} = \frac{2\pi R}{T}\left[-\sin(\frac{2\pi}{T}t)\right]\frac{2\pi}{T} = -\frac{4\pi^2 R}{T^2}\sin(\frac{2\pi}{T}t)$$

Now using the Pythagorean Theorem again, we find the net (Centripetal Acceleration) as follows:

$$a_c = \sqrt{a_x{}^2 + a_y{}^2} = \sqrt{\left[-\frac{4\pi^2 R}{T^2}\cos(\frac{2\pi}{T}t)\right]^2 + \left[-\frac{4\pi^2 R}{T^2}\sin(\frac{2\pi}{T}t)\right]^2} = \sqrt{\left[-\frac{4\pi^2 R}{T^2}\right]^2\left(\sin^2(\frac{2\pi}{T}t) + \cos^2(\frac{2\pi}{T}t)\right)}$$

$$= \sqrt{\left[-\frac{4\pi^2 R}{T^2}\right]^2(1)} = \frac{4\pi^2 R}{T^2}$$

Note that this is the same equation as way above. The only difference is that we derived it from basic physics and Calculus without the usual 'hand waving' we used in Physics 1.

Just one of MANY examples you will see during the year of the power of Calculus.

Now we have to address interlocking functions that appear from time to time in this course. They are the exponential functions given by:

$y = a^x$ *and* $y = e^x$ and their INVERSE functions given by: $y = \log_a(x)$ *and* $y = \ln(x)$

We will derive the derivatives using the same computer approximation as we did for sine and cosine. We will do $y = a^x$ first and then extend it to $y = 10^x$ and $y = e^x$.

$y = a^x$

$$y = a^x\text{, therefore } \frac{dy}{dx} = \frac{a^{x+\Delta x} - a^x}{\Delta x} = \frac{a^x a^{\Delta x} - a^x}{\Delta x} = a^x \frac{a^{\Delta x} - 1}{\Delta x}$$

Now, just what is that strange function $\frac{a^{\Delta x} - 1}{\Delta x}$? Using Excel once again let's play with it (this is also a live spreadsheet.)

a=	2				a=	100		
Δx	$a^{\Delta x}$	$a^{\Delta x}$-1	Ratio		Δx	$a^{\Delta x}$	$a^{\Delta x}$-1	Ratio
1.000000000	2.0000000	1.0000000	1.0000000		1.000000000	100.0000000	99.0000000	99.0000000
0.100000000	1.0717735	0.0717735	0.7177346		0.100000000	1.5848932	0.5848932	5.8489319
0.010000000	1.0069556	0.0069556	0.6955550		0.010000000	1.0471285	0.0471285	4.7128548
0.001000000	1.0006934	0.0006934	0.6933875		0.001000000	1.0046158	0.0046158	4.6157903
0.000100000	1.0000693	0.0000693	0.6931712		0.000100000	1.0004606	0.0004606	4.6062307
0.000010000	1.0000069	0.0000069	0.6931496		0.000010000	1.0000461	0.0000461	4.6052762
0.000001000	1.0000007	0.0000007	0.6931474		0.000001000	1.0000046	0.0000046	4.6051808
0.000000100	1.0000001	0.0000001	0.6931472		0.000000100	1.0000005	0.0000005	4.6051712
0.000000010	1.0000000	0.0000000	0.6931472		0.000000010	1.0000000	0.0000000	4.6051703
0.000000001	1.0000000	0.0000000	0.6931471		0.000000001	1.0000000	0.0000000	4.6051702
		Ln(2)=	0.6931472				Ln(100)=	4.6051702
a=	10				a=	143.296		
Δx	$a^{\Delta x}$	$a^{\Delta x}$-1	Ratio		Δx	$a^{\Delta x}$	$a^{\Delta x}$-1	Ratio
1.000000000	10.0000000	9.0000000	9.0000000		1.000000000	143.2960000	142.2960000	142.2960000
0.100000000	1.2589254	0.2589254	2.5892541		0.100000000	1.6429464	0.6429464	6.4294644
0.010000000	1.0232930	0.0232930	2.3292992		0.010000000	1.0509023	0.0509023	5.0902296
0.001000000	1.0023052	0.0023052	2.3052381		0.001000000	1.0049773	0.0049773	4.9772580
0.000100000	1.0002303	0.0002303	2.3028502		0.000100000	1.0004966	0.0004966	4.9661451
0.000010000	1.0000230	0.0000230	2.3026116		0.000010000	1.0000497	0.0000497	4.9650357
0.000001000	1.0000023	0.0000023	2.3025877		0.000001000	1.0000050	0.0000050	4.9649247
0.000000100	1.0000002	0.0000002	2.3025854		0.000000100	1.0000005	0.0000005	4.9649137
0.000000010	1.0000000	0.0000000	2.3025851		0.000000010	1.0000000	0.0000000	4.9649125
0.000000001	1.0000000	0.0000000	2.3025850		0.000000001	1.0000000	0.0000000	4.9649125
		Ln(10)=	2.3025851				Ln(143.296)=	4.9649124

As you can see, the strange little fraction $\frac{a^{\Delta x}-1}{\Delta x}$ approaches $\ln(a)$ as $\Delta x \to 0$.

We suggest you open the spreadsheet and play with the values of 'a' to see what happens.

Thus the derivative of $y = a^x$ is simply $a^x \ln a$ for any positive a.

As you might guess, the above table(s) are NOT proofs, they are demonstrations. I leave the hard job of proving all this to the Calculus teachers

For Base 10 we have

$$y = 10^x$$

Thus

$$\frac{dy}{dx} = 10^x \ln(10) = 2.3025851 * 10^x$$

$$Also, for \quad y = e^x$$

$$\frac{dy}{dx} = e^x \ln(e) = 1 * e^x = e^x \quad \text{(Recall that } \ln(e) = 1\text{)}$$

This little derivative fully explains why $y = e^x$ is so common in upper math and science courses. The derivative is the same as the original function. Sure doesn't take much work to do this derivative.

$y = \ln(x)$

Last, but not least, we will determine the derivative of $y = ln(x)$.

Given

$$y = ln(x)$$

$$y' = \frac{dy}{dx} = \frac{ln(x+\Delta x) - ln(x)}{\Delta x}$$

Remember your logarithm manipulation rules!

Given

$$y = ln(x)$$

$$y' = \frac{dy}{dx} = \frac{ln(x+\Delta x) - ln(x)}{\Delta x} = \frac{1}{\Delta x} ln\left[\frac{x+\Delta x}{x}\right] = \frac{1}{\Delta x} ln\left[1+\frac{\Delta x}{x}\right]$$

$$= \frac{x}{x}\frac{1}{\Delta x} ln\left[1+\frac{\Delta x}{x}\right] = \frac{1}{x}\frac{x}{\Delta x} ln\left[1+\frac{\Delta x}{x}\right] = \frac{1}{x} ln\left[1+\frac{\Delta x}{x}\right]^{\frac{x}{\Delta x}}$$

Now, what does one do with the expression $\left[1+\frac{\Delta x}{x}\right]^{\frac{x}{\Delta x}}$?

Well, once again we go to Excel and just let it play with the numbers and see what happens. BTW:

$$Mess = \left[1 + \frac{\Delta x}{x}\right]^{\frac{x}{\Delta x}}$$

Note that in every case above, the expression $\left[1 + \frac{\Delta x}{x}\right]^{\frac{x}{\Delta x}}$ regardless of the value of x, always becomes 2.71823… which is better known as ‘e’. Thus we have

x=	1	
Δx	Δx/x	Mess
1.0000000	1.0000000000	2.0000000
0.1000000	0.1000000000	2.5937425
0.0100000	0.0100000000	2.7048138
0.0010000	0.0010000000	2.7169239
0.0001000	0.0001000000	2.7181459
0.0000100	0.0000100000	2.7182682
0.0000010	0.0000010000	2.7182805
0.0000001	0.0000001000	2.7182817
0.0000000	0.0000000100	2.7182818
0.0000000	0.0000000010	2.7182821

x=	100	
Δx	Δx/x	Mess
1.0000000	0.01000000000	2.7048138
0.1000000	0.00100000000	2.7169239
0.0100000	0.00010000000	2.7181459
0.0010000	0.00001000000	2.7182682
0.0001000	0.00000100000	2.7182805
0.0000100	0.00000010000	2.7182817
0.0000010	0.00000001000	2.7182818
0.0000001	0.00000000100	2.7182821
0.0000000	0.00000000010	2.7182821
0.0000000	0.00000000001	2.7182821

x=	10	
Δx	Δx/x	Mess
1.0000000	0.1000000000	2.5937425
0.1000000	0.0100000000	2.7048138
0.0100000	0.0010000000	2.7169239
0.0010000	0.0001000000	2.7181459
0.0001000	0.0000100000	2.7182682
0.0000100	0.0000010000	2.7182805
0.0000010	0.0000001000	2.7182817
0.0000001	0.0000000100	2.7182818
0.0000000	0.0000000010	2.7182821
0.0000000	0.0000000001	2.7182821

x=	0.1	
Δx	Δx/x	Mess
1.0000000	10.00000000000	1.2709816
0.1000000	1.00000000000	2.0000000
0.0100000	0.10000000000	2.5937425
0.0010000	0.01000000000	2.7048138
0.0001000	0.00100000000	2.7169239
0.0000100	0.00010000000	2.7181459
0.0000010	0.00001000000	2.7182682
0.0000001	0.00000100000	2.7182805
0.0000000	0.00000010000	2.7182817
0.0000000	0.00000001000	2.7182818

Given

$$y = \ln(x)$$

$$y' = \frac{dy}{dx} = \frac{ln(x+\Delta x) - ln(x)}{\Delta x} = \frac{1}{\Delta x} ln\left[\frac{x+\Delta x}{x}\right] = \frac{1}{\Delta x} ln\left[1+\frac{\Delta x}{x}\right]$$

$$= \frac{x}{x}\frac{1}{\Delta x} ln\left[1+\frac{\Delta x}{x}\right] = \frac{1}{x}\frac{x}{\Delta x} ln\left[1+\frac{\Delta x}{x}\right] = \frac{1}{x} ln\left[1+\frac{\Delta x}{x}\right]^{\frac{x}{\Delta x}} = \frac{1}{x} ln(e) = \frac{1}{x}$$

Thus the derivative of $y = ln(x)$ is simply $y' = \frac{dy}{dx} = 1/x$. Again, this is a very simple result.

This material represents all of the derivative information you need for the AP-C Physics course.

However, you just knew there was a however! It is only ***half*** of the calculus we will use.

Integral Calculus

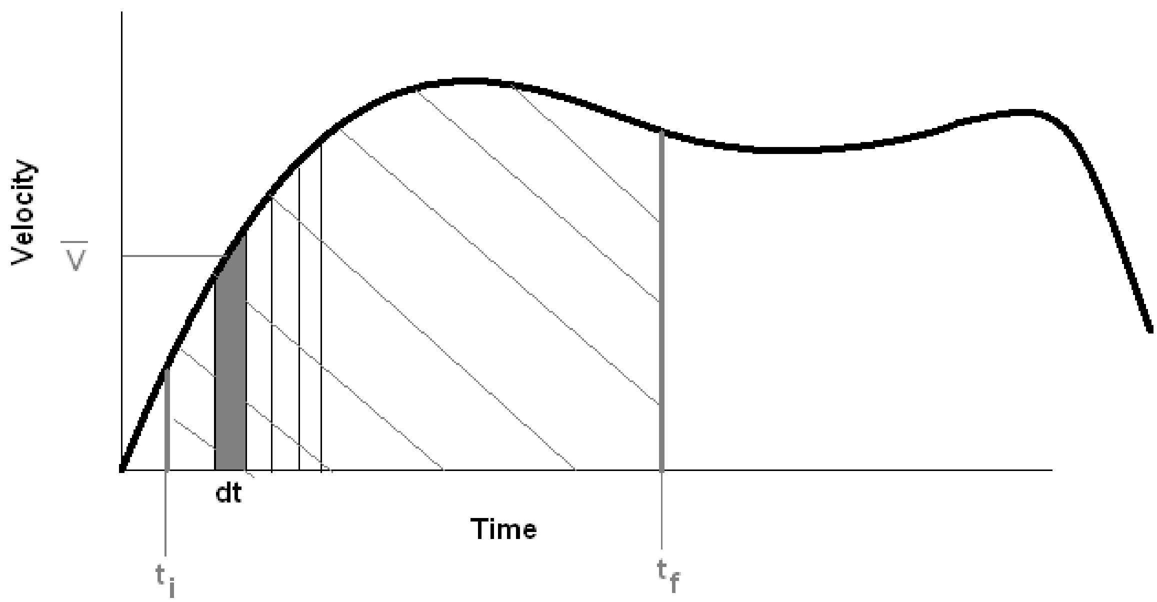
There is a strong connection between finding slopes of curves and finding the area under the curve.

Without doing the entire fancy math done in a REAL Calculus class let's just list what we learned last year.

1. The *slope* of a position-time curve is velocity.
2. The *slope* of a velocity-time curve is acceleration.
3. The *area* under a velocity-time curve is change in position.
4. The *area* under an acceleration-time curve is the change in velocity.

Suppose we have the equation $V = \frac{dx}{dt}$ which just says that the velocity is the slope of position graph.

We can rewrite the equation is $dx = Vdt$



In the graph, the gray shaded area is approximately a trapezoid with an average height of $\bar{V}$ better known as the average velocity. If the interval is very small the Average Velocity will simply be the average of the end points. The area beneath that segment of the curve will tell us how far we traveled during the interval dt. The area is simply Vdt which is also dx from above. Thus the TOTAL change in 'x' is simply the sum of the little changes in each little area of width dt.

But, the change in x can also be computed by simply subtracting the initial position from the final position.

What we have is the sum of all the Vdt's from t_i to t_f will give us the TOTAL $dx = x_f - x_i$.

Now, suppose we have more than just data. We have an actual function that represents the velocity curve.

Suppose $v = f(t)$.

We already know that this function $f(t)$ is a slope of a position function (last year). We also know how to find slopes. Thus, we need to UNSLOPE $v = f(t)$. We need to find a function that has as its slope, $f(t)$.

Examples

Suppose $v = 2t$. What function when differentiated gives $v = 2t$? It doesn't take too much thought to realize it is $x = t^2$. (Just differentiate the second equation.)

Thus we know that $x = t^2$will get the job done.

If $x = t^2$, the $dx/dt = 2t = v$. However, there are actually infinitely many such equations as we will see.

Suppose $x = t^2 + 3$, then $dx/dt = 2t = v$, also. In fact the 3 could be *any* constant and it would still work. What we do is simply add a C (constant) to the UNSLOPE to get the general equation.

So, if $v = 2t$ then $x = t^2 + C$.

Now, going back up, we know that the total area under the curve between t_i and t_f is the total dx for the curve but we also know that $dx = x_f - x_i$.

This allows us to determine the area under the curve by simply computing x_f and x_i and then subtracting them.

In our case above, $x = t^2 + C$ which means that $x_f = t_f^2 + C$ and $x_i = t_i^2 + C$.

Total $dx = x_f - x_i = \left(t_f^2 + C\right) - (t_i^2 + C) = t_f^2 - t_i^2$.

What we have done is simply find the ANTIDERIVATIVE of the function and plug in the endpoints and subtract. The constant 'C' is called the Constant of Integration and depends on knowing the values of the variables in the Integral at some point.

But we are also adding up infinitely many minute area slices under the curve which is called integration. We are really finding a sum, which is why the symbol for integration is $\int$ (see the 's' shape?). If $V = \frac{dx}{dt}$, then $dx = Vdt$, and $x = \int dx = \int Vdt$. This is the notation for an indefinite integral.

INTEGRAL is a fancy word for anti-derivative which is a fancy word for UNSLOPE which is an ugly word meaning undo the derivative you did to make the function we are now trying to undo (READ THAT SEVERAL TIMES – FAST).

Thus, a curve's definite integral is the difference in area under the curve between two points on the x-axis.

This is known as the **Fundamental Theorem of Calculus.**

Examples

1. Find the area under $y = 3x^2$ between $x = 1$ and $x = 5$. (i.e. find the integral of $y = 3x^2$ or $\int_1^5 3x^2 dx$)

    The Integral (anti-derivative = unslope) is $Y = x^3 + C$ where $Y' = y$.
    Computing $Y(5) - Y(1) = (125 + C) - (1 + C) = 124$

2. If velocity is given by $v = 4t^3 + t^2 + 2$, how far do I travel between $t = 0$ and $t = 2$? (i.e. find the integral of $v = 4t^3 + t^2 + 2$ or $x = \int_0^2 (4t^3 + t^2 + 2)dt$)

    The integral is $\int_0^2 (4t^3 + t^2 + 2)dt = t^4 + (1/3)t^3 + 2t + C$.

    The area is $x(2) - x(0) = (24 + (\frac{1}{3})23 + 2(2) + C) - (0 + 0 + 0 + C) = (16 + \frac{8}{3} + 4 + C) - (C) = 22\frac{2}{3}$

3. What is the area under a sine curve from $x = 0$ to $x = \pi/2$?

   Find the integral of $y = sin(x)$. The slope of $cos(x) = -sin(x)$ therefore the integral must be $Y = -cos(x) + C$.

   Plug in the endpoints and $Y = (-cos(\pi/2) + C) - (-cos(0) + C) = (-0 + C) - (-1 + C) = 1$.

   Thus the area under a half hump from 0 to $\pi/2$ must be1!

4. Suppose a force that is applied to a car is a function of the car's position. How much work must be done to move the car from $x = 2$ to $x = 4$ if the force function is $F(x) = 300x$? (Newtons)

   This could easily be accomplished by a strong spring. In fact, we did this last year when we studied springs. The work done is simply the area under a Force v. Distance curve.

   Since we need an area, just find the integral.

   $\text{AREA} = \text{Integral of } F(x) = 150x^2 + C$. Do you remember the formula for spring energy? $\left[PE_e e = (\frac{1}{2})kx^2\right]$

   Plug in the endpoints and we get $\text{AREA} = 150(4)^2 + C) - (150(2)^2 + C) = 150(12) = 1800 \text{ Joules}$.

5. Suppose we have the following information: $V = 5t^2$ and $x = 3$ at $t = 2$. What equation represents the object's acceleration and what equation represents the object's position?

   The acceleration is simply the derivative of the velocity, thus $a = \frac{dv}{dt} = 10t$, and we are done.

   The position is the integral, which is $x = (\frac{5}{3})t^3 + C$; however, we know that $x = 3$ when $t = 2$. Using that information, we solve for C, and get $3 = (\frac{5}{3})2^3 + C$, or $C = 3 - (\frac{5}{3})8 = -10.33$. The position equation is $x = (5/3)t^3 - 10.33$.

6. Suppose we cause a car to accelerate constantly at a and require that $v = v_i$, and $x = x_i$ at $t = 0$. What will be the car's velocity and position at some later time t?

   The relationship between acceleration and velocity is the same as the relationship between velocity and position. Thus, to find velocity we simply integrate the acceleration.

   Given that $\text{Acc} = a$, the integral MUST be $v = at + C$. But, at $t = 0$, $v = v_i$. Therefore, $v_i = a(0) + C$, or $C = v_i$. Therefore $v = at + v_i$. This is the same equation we learned last year.

   To find position, we just integrate velocity ($v = at + v_i$) which gives $x = \left(\frac{1}{2}\right)at^2 + v_i t + C$. (Note: this '$C$' is a different '$C$' than the one in the velocity equation above). Therefore $x_i = \left(\frac{1}{2}\right)a(0)^2 + v_i(0) + C$ or $C = x_i$. The final function is $x_f = \left(\frac{1}{2}\right)at^2 + v_i t + x_i$. This also is the same as the equation from last year.

7. Demonstrate that the anti-derivative is the same as the area under the curve. Suppose an object moves in such a way that its velocity is given by $v = 2t^3 + 4t + 2$. How far would it travel between $t = 1\text{s}$ and $t = 3\text{s}$?

   Using Calculus, we simply find the integral (anti-derivative, unslope) and plug in the x-axis end points and subtract, $x = (1/4)(2)t^4 + 2t^2 + 2t + C$. Computing, $x(3) - x(1)$ we get,

$$dx = \left[(\frac{1}{2})(3^4) + 2(3^2) + 2(3) + C \right] - \left[(\frac{1}{2})(1^4) + 2(1^2) + 2(1) + C \right]$$

$$= \left[\frac{1}{2}(81) + 2(9) + 6 + C \right] - \left[(\frac{1}{2})(1) + 2(1) + 2(1) + C \right] = \left[40.5 + 18 + 6 + C \right] - \left[0.5 + 4 + C \right] = 60$$

This is so simple (on the surface!)

Now, I am going to use a special mathematical method to find the actual areas by adding up a whole bunch of small trapezoids under the curve.

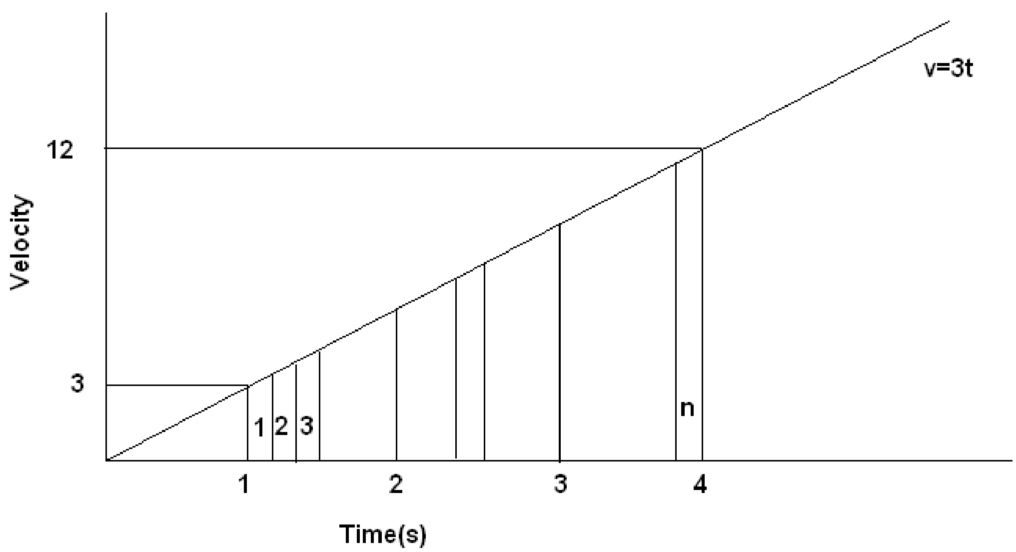
Suppose we have the function $v = 3t$. What is the change in position between $t = 1\text{s}$ and $t = 4\text{s}$?

Using what we have just seen, we integrate the function, plug in the end points and subtract.

$x = (\frac{3}{2})t^2 + C$. The change in x from $t = 1$ to $t = 4$ is simply $dx = ((\frac{3}{2}) 4^2 + C) – ((\frac{3}{2}) 1^2 + C)$

or

$dx = \left(\frac{3}{2}\right) 16 - \left(\frac{3}{2}\right) 1 = \left(\frac{3}{2}\right) 15 = 45/2 = 22.5$



Now, let's actually compute the area by adding up little trapezoids.

The graph above shows the velocity curve, $v = 3t$, and I have added some sample trapezoids between $t = 1\text{s}$ and $t = 4\text{s}$. Assume there are 'n' such trapezoids from $t = 1\text{s}$ to $t = 4\text{s}$.

If the gaps are all the same width, then each dt is $\frac{(4-1)}{n} = \frac{3}{n}$ wide. The average height of each trapezoid is approximately the height or the right edge of the trapezoid. (We could just as easily use the left edge or the average height.)

The right edge of trapezoid #1 ends at $t = 1 + 1 \cdot \left(\frac{3}{n}\right)$. Thus the height is simply $v = 3\left(1 + 1 \cdot \left(\frac{3}{n}\right)\right)$.

The area of the trapezoid is $\text{height} \cdot \text{width} = v \cdot dt = 3\left(1 + 1 \cdot \left(\frac{3}{n}\right)\right) \cdot \frac{3}{n}$.

The right edge of trapezoid #2 ends at $t = 1 + 2 \cdot \left(\frac{3}{n}\right)$. Thus the height is $v = 3\left(1 + 2 \cdot \left(\frac{3}{n}\right)\right)$.
The area is $3\left(1 + 2 \cdot \left(\frac{3}{n}\right)\right) \cdot \frac{3}{n}$.

The right edge of trapezoid #3 ends at $t = 1 + 3 \cdot \left(\frac{3}{n}\right)$. Thus the height is $v = 3\left(1 + 3 \cdot \left(\frac{3}{n}\right)\right)$.
The area is $3\left(1 + 3 \cdot \left(\frac{3}{n}\right)\right) \cdot \frac{3}{n}$.

And so on…

The right edge of the n^{th} trapezoid ends at $t = 1 + n \cdot \left(\frac{3}{n}\right) = 4$ (dah!). The height is $3\left(1 + n \cdot \left(\frac{3}{n}\right)\right)$.
The area is $3\left(1 + n \cdot \left(\frac{3}{n}\right)\right) \cdot \frac{3}{n}$.

Now the total area (approximated) is

$$\text{AREA} = 3\left(1 + 1 \cdot \left(\frac{3}{n}\right)\right) \cdot \frac{3}{n} + 3\left(1 + 2 \cdot \left(\frac{3}{n}\right)\right) \cdot \frac{3}{n} + 3\left(1 + 3 \cdot \left(\frac{3}{n}\right)\right) \cdot \frac{3}{n} + \cdots + 3\left(1 + n \cdot \left(\frac{3}{n}\right)\right) \cdot \frac{3}{n}.$$

Now, using RULE 1 from page 2, expand this expression.

$$\text{AREA} = \frac{9}{n} + \left(1\left(\frac{27}{n^2}\right)\right) + \frac{9}{n} + \left(2 \cdot \left(\frac{27}{n^2}\right)\right) + \frac{9}{n} + \left(3 \cdot \left(\frac{27}{n^2}\right)\right) + \cdots + \frac{9}{n} + \left(n \cdot \left(\frac{27}{n^2}\right)\right)$$

or

$$\text{AREA} = \left(\frac{9}{n} + \frac{9}{n} + \frac{9}{n} + \cdots + \frac{9}{n}\right) + \left(1\left(\frac{27}{n^2}\right)\right) + \left(2 \cdot \left(\frac{27}{n^2}\right)\right) + \left(3 \cdot \left(\frac{27}{n^2}\right)\right) + \cdots + \left(n \cdot \left(\frac{27}{n^2}\right)\right)$$

or

$$\text{AREA} = 9\left(\frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}\right) + \left(\frac{27}{n^2}\right) \cdot (1 + 2 + 3 + \cdots + n).$$

Since there are n rectangles, adding up the same number n times gives $n \cdot (\text{the number})$. In our case the same number is $(1/n)$ thus $n\left(\frac{1}{n}\right) = 1$, giving

$$\text{AREA} = 9 * (1) + \left(\frac{27}{n^2}\right) \cdot (1 + 2 + 3 + \cdots + n).$$

Now is there a nifty expression, $(1 + 2 + 3 + \cdots + n)$?!

Suppose $S = 1 + 2 + 3 + \cdots + n$
Also, $S = n + n - 1 + n - 2 + \cdots + 1$
When added, $2S = (n + 1) + (n + 1) + (n + 1) + \cdots + (n + 1) = n(n + 1)$.

Therefore $S = \left(\frac{1}{2}\right)n * (n + 1)$!

Thus

$$\text{AREA} = 9 * (1) + \left(\frac{27}{n^2}\right) * \left(\frac{1}{2}\right) n(n + 1)$$

or

$$\text{AREA} = 9 + \left(\frac{27}{n^2}\right)\left(\frac{1}{2}\right)(n^2 + n) = 9 + \left(\frac{13.5}{n^2}\right)(n^2 + n) = 9 + 13.5 + \frac{13.5}{n} = 22.5 + 13.5/n.$$

If we let $n \to \infty$ then $13.5/n \to \infty$ and we have $\text{AREA} = 22.5$, which is the same as what we obtained previously, using Calculus.

This should convince you that we don't really want to calculate areas this way. However, it is actually rather common when the function to be integrated is too hard to integrate symbolically.

BTW: In this particular case we could have simply calculated the area since the ORIGINAL area is itself a simple trapezoid.

$\text{AREA} = (\text{Average Height}) * (\text{Base}) = (3 + 12)/2 * 3 = (15/2) * 3 = 45/2 = 22.5.$

What you ARE seeing is that Calculus is ready-made to make Physics easier (if and only if, one can use functional equations instead of discrete data points as we did last year). This means that there is going to be a strong reliance on fitting functions to data in this class. Of course we also did it last year with linear regression. We will continue with linear regression and may add polynomial and log-log curve fits as the year progresses.

Simply stated, if a function represents a derivative, then both the area under that derivative and the function that produced the derivative (the Integral) are obtained by essentially the same process.

What you are NOT seeing is the HARD fact that the act of Integration is far more difficult than the act of differentiation. Differentiation is taught in a week. Integration takes multiple courses and you have only begun.